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# Notes and Correspondence

# Comments on deriving the equilibrium height of the stable boundary layer

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**ABSTRACT:** Recently, the equilibrium height of the stable boundary layer received much attention in a series of papers by Zilitinkevich and co-workers. In these studies the stable boundary-layer height is derived in terms of inverse interpolation of different boundary-layer height scales, each representing a prototype boundary layer. As an alternative we propose an inverse interpolation of the eddy diffusivities for each prototype before applying the definition of the Ekman layer depth. The new equation for the stable boundary-layer height improves performance in a comparison against four observational datasets. Copyright © 2007 Royal Meteorological Society

KEY WORDS Ekman layer; mixing height; stable boundary layer

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## 1. Introduction

The equilibrium height of the stable boundary layer, $h_E$, and its relevance for predicting the stable boundary-layer (SBL) structure and for air-quality modelling, has been discussed intensively (by, among others, Zilitinkevich and Esau, 2003, henceforth ZE03; Steeneveld *et al.*, 2006). Recently, several papers (Zilitinkevich and Mironov, 1996; Zilitinkevich and Calanca, 2000; Zilitinkevich and Baklanov, 2002; ZE03) discuss the relevant processes that govern the stable boundary-layer height in equilibrium conditions. In these studies, the basic variables governing $h_E$ are the surface friction velocity $u_*$, the surface buoyancy flux $B_s = g\overline{w'\theta'}/\overline{\theta}$, the Coriolis parameter $f$ and the free-flow stability $N$. ($g$, $\theta$ and $w$ are acceleration due to gravity, the potential temperature and vertical velocity, respectively.) Based on these variables, ZE03 identified three boundary-layer prototypes: the truly neutral ($B_s = 0$ and $N = 0$), the conventionally neutral ($N \neq 0$ and $B_s = 0$) and the nocturnal boundary layer ($N = 0$ and $B_s \neq 0$).

In the papers by Zilitinkevich and co-workers, the coupling of these prototypes is done by interpolation of associated boundary-layer height-scales. In this paper we propose an approach directly related to the bulk eddy diffusivity of the prototypes. We will show that the new alternative improves predictive skill.

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## 2. Background

Following the reasoning by ZE03, the stable boundary-layer height is defined as the Ekman layer depth ($h_*$), which is given by a bulk value for the eddy diffusivity $K_M$ and the absolute value of the Coriolis parameter $f$ (e.g. Stull, 1988):

$$h_* = \sqrt{\frac{K_M}{f}}. \quad (1)$$

For the eddy viscosity $K_M$, ZE03 distinguish three different boundary-layer types, and for each type a characteristic velocity-scale $u_T$ and length-scale $l_T$ are defined as follows:

$$\text{Truly neutral} \quad K_M = u_T l_T = u_* h_*, \quad (2)$$

$$\text{Conventionally neutral} \quad K_M = u_T l_T = \frac{u_*^2}{N}, \quad (3)$$

$$\text{Nocturnal} \quad K_M = u_T l_T = u_* L. \quad (4)$$

Here $L = -u_*^3/B_s$ is the Obukhov length. (Note that the von Kármán constant is not included here.) ZE03 obtain an equilibrium height for each boundary-layer prototype:

$$\text{Truly neutral} \quad h_{E,TN} = C_R \frac{u_*}{f}, \quad (5)$$

$$\text{Conventionally neutral} \quad h_{E,CN} = \frac{C_S}{\sqrt{C_{uN}}} \frac{u_*}{\sqrt{|fN|}}, \quad (6)$$

$$\text{Nocturnal} \quad h_{E,Noct} = C_S \frac{u_*^2}{\sqrt{|f B_s|}}. \quad (7)$$

To obtain an equilibrium height $h_E$ that accounts for all three combined prototypes, the equilibrium heights of the individual prototypes are interpolated as follows:

$$\frac{1}{h_E^2} = \frac{1}{h_{E,TN}^2} + \frac{1}{h_{E,CN}^2} + \frac{1}{h_{E,Noct}^2}. \tag{8}$$

Then

$$h_E = C_R \frac{u_*}{f} \left( 1 + \frac{C_R^2 C_{uN}}{C_S^2} \frac{N}{f} + \frac{C_R^2}{C_S^2} \frac{u_*}{fL} \right)^{-\frac{1}{2}}. \tag{9}$$

Here $f \neq 0$ and $C_R = 0.5$, $C_{uN}/C_S^2 = 0.56$, $C_S = 1.0$ are dimensionless empirical constants.

If the relevant eddy diffusivities are indeed well represented by Equations (2)–(4), we note that the bulk diffusivity $K_M$ directly can be written as

$$\frac{1}{K_M} = \frac{1}{u_* h_*} + \frac{1}{u_*^2/N} + \frac{1}{u_* L}. \tag{10}$$

Here the proportionality coefficients are taken equal to 1 for convenience. Consequently

$$K_M = \frac{u_*^2 h_* L/N}{(u_* h_*/N) + h_* L + (u_* L/N)}. \tag{11}$$

Combining Equations (11) and (1), solving for $h_* = h_E$ and choosing the physical solution in the quadratic equation, we obtain:

$$h_E = \alpha \frac{u_*}{N}, \tag{12}$$

where

$$\alpha = \frac{-1 + \sqrt{1 + 4\left(\frac{u_*}{fL} + \frac{N}{f}\right)}}{2\left(\frac{u_*}{NL} + 1\right)}. \tag{13}$$

Also Equation (9) can be written in the format of Equation (12). Then $\alpha$ is given by:

$$\alpha = \frac{C_R N/f}{\left( 1 + \frac{C_R^2 C_{uN}}{C_S^2} \frac{N}{f} + \frac{C_R^2}{C_S^2} \frac{u_*}{fL} \right)^{1/2}}, \tag{14}$$

which is clearly different from Equation (13).

The format of Equation (12) was already found in many studies. Vogelezang and Holtslag (1996) found $\alpha$ to be a function of the shear and Richardson number across the SBL, while Steeneveld *et al.* (2006) derived Equation (12) with $\alpha$ solely depending on the free-flow stability. In any case, Equations (13) and (14) show that $\alpha$ is related to the traditional parameter groups $u_*/(fL)$ (the Monin–Kazanski parameter) and $N/f$ (Kitaigordskii and Joffre, 1988). The numerical value of $\alpha$ is typically 7 to 13 (e.g. Vogelezang and Holtslag, 1996).

## 3. Observations and results

In order to validate Equation (12) with Equation (13), and to compare its performance with Equation (9), we use the dataset described in Steeneveld *et al.* (2006). The dataset consists of observed SBL heights, turbulent surface fluxes (eddy covariance) and free-flow stability over a wide range of latitude, surface roughness ($z_0$) and land use. Data are available from Cabauw (149 data points, $z_0 = 0.20$ m, grassland, 51 °N, The Netherlands), Sodankyla (30 data points, $z_0 = 1.4$ m, boreal forest, 67 °N, Finland), CASES-99 (32 data points, $z_0 = 0.03$ m, prairie grassland, 37 °N, USA) and SHEBA (20 data points, $z_0 = 1.10^{-4}$ m, sea ice, 75 °N). The SBL height was obtained from soundings using the method in Joffre *et al.* (2001), except for Cabauw where $h$ was obtained from sodar measurements. The observations have been selected for $u_* > 0.04$ m s$^{-1}$, $\overline{w'\theta'} < -0.0016$ K m s$^{-1}$ and $N > 0.015$ s$^{-1}$ to ensure a reliable dataset. For more details see Steeneveld *et al.* (2006).

Results obtained with Equation (9) and with Equation (12) with (13) are shown in Figures 1 and 2, respectively. Table I summarizes some statistical quantities for model performance, i.e. mean absolute error (MAE), systematic RMSE (RMSE-S), median of the mean absolute error (MEAE) and the index of agreement (IoA, Willmott 1982; the IoA equals 1 for a perfect model performance). Equation (12) gives a substantial reduction of the RMSE-S, and an increased IoA compared to Equation (9). Note that for shallow SBLs, mesoscale effects may become important and these may contribute to the bias, since mesoscale effects are not incorporated in the

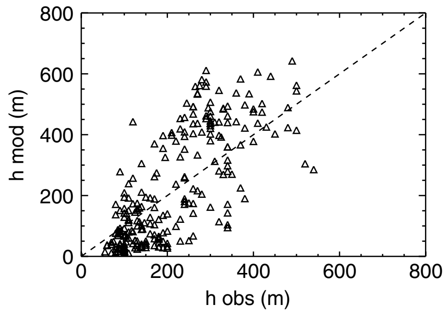


Figure 1. Modelled (Equation (9)) versus observed stable boundary-layer height.

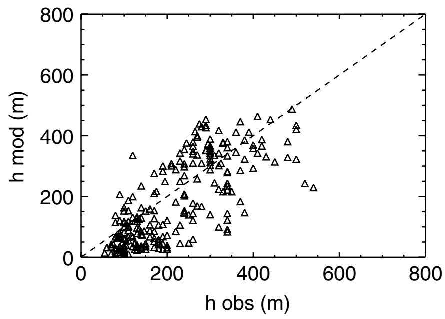


Figure 2. Modelled (Equation (12)) versus observed stable boundary-layer height.



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Table I. Statistical evaluation of SBL height proposals.

| Model | MAE (m) | RMSE-S (m) | MEAE (m) | IoA |
|---|---|---|---|---|
| Equation (9) | 100.9 | 99.3 | 83.0 | 0.80 |
| Equation (12) | 78.7 | 62.9 | 67.9 | 0.84 |
| Equation (15) | 65.2 | 41.5 | 49.7 | 0.84 |

See text for acronyms

current model. Unfortunately the proposed interpolation method cannot avoid the negative bias for shallow SBLs.

As an alternative, Steeneveld *et al.* (2006) applied a formal dimensional analysis on $h$, $u_*$, $N$, and $B_s$, not taking into account $f$. See Steeneveld *et al.* (2006) for discussion of the relevance of this parameter. This gives the dimensionless groups $\Pi_1 = hN/u_*$ and $\Pi_2 = h/L$. Then it is found that the equilibrium SBL height is given by

$$h_E = 10\frac{u_*}{N}, \qquad \text{for } \frac{u_*^2 N}{|B_s|} > 10, \tag{15a}$$

$$h_E = 31.6\sqrt{\frac{|B_s|}{N^3}}, \qquad \text{for } \frac{u_*^2 N}{|B_s|} \leq 10. \tag{15b}$$

Figure 3 shows that the negative bias for a shallow SBL is not present with Equations (15), in particular due to the impact of Equation (15b) for (very) stable conditions. For moderately stable and near neutral conditions, Equation (15a) does also well, even with a constant value of the coefficient (here 10). Thus a satisfactory prediction of the SBL height can be obtained without taking into account $f$ explicitly (see also discussion in Vogelezang and Holtslag, 1996). Note that this formula is only valid for the range of the variables for which it has been derived. Nevertheless it is worthwhile considering its applicability beyond this range, i.e. its limit behaviour. The formula behaves properly for $B_s \to 0$, since in this case the upper branch should be utilized. For $N \to 0$, Equation (15) seems not *a priori* to approach a proper limit. Formally speaking, Equations (15) would lead to unrealistically deep SBLs. However, in practice, this limit is hardly ever found in the atmosphere due to radiation divergence, which depends on the temperature profile rather than on the potential temperature profile. Finally, we note that Equation (9) has a proper limit behaviour for $N \to 0$, and $B_s \to 0$, but this in turn leads to an infinite SBL depth for $f \to 0$ (the equatorial case).

We realize that the evaluation of the above equations for the equilibrium depth with field data may be troublesome, due to the complexity of making observations in stable conditions and the fact that in reality conditions cannot be controlled. Alternatively, we may consider to explore large-eddy simulation (LES) results for more controlled testing (as in Esau, 2004). In that case however, we must be aware of the fact that, especially in very stable conditions, LES results (profiles of mean and turbulent quantities) are strongly dependent on the model

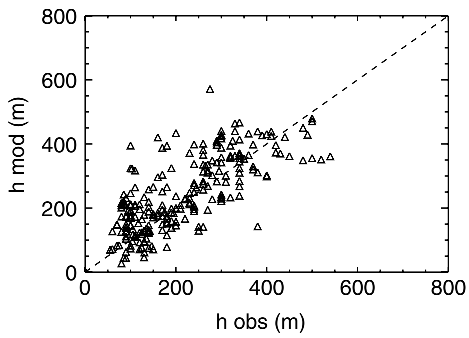


Figure 3. Modelled (Equation (15)) versus observed stable boundary-layer height.

resolution (Beare and MacVean, 2004). Also long-wave radiation divergence plays an important role, which is usually not taken into account by LESs. Note that the field data used in this study cover a wide range of conditions, including non-turbulent effects such as radiation divergence (e.g. André and Mahrt, 1982).

## 4. Conclusions

We propose an alternative method to derive a formula for the stable boundary-layer height when more than one stable boundary-layer prototype contributes to the final boundary-layer height. Instead of interpolating the height scales for each prototype, we directly interpolate the eddy diffusivities of each prototype. The alternative formulation performs well, and reduces the bias of the predicted stable boundary-layer height compared to the original formulation. Furthermore, a second alternative based on formal dimensional analysis shows improved skill, especially for shallow stable boundary layers. Further improvements are possible by following the full approach in Steeneveld *et al.* (2006).

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